GET THE RIGHT START SOLVING ALGEBRA WORD PROBLEMS

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1. INTRODUCTION

During all my years of tutoring mathematics students, I found that word problems were most upsetting for students. A problem would baffle students who had little trouble with the equations involved, simply because words were included. Unfortunately, when the student asks for help, the parent or friend probably experienced the same mental block when he or she encountered word problems in school.

In many occupations and in higher mathematics, word problems predominate. If a student looks forward to advancing in school and securing a good job after graduation, word problems often must be mastered. Also, once word problems are mastered, the student will find them much more interesting than solving equations alone.

1.1 Best Learning Method

I have found that the Socratic method works best in learning most subjects. Mathematics is no exception. The Socratic method breaks problems down into a series of questions the student is asked to answer. These questions and answers are designed to lead to a solution of the problem. The student's responses at any stage of the solution may be incorrect. If that happens, the tutor needs to break down the question into several simpler questions after suggesting why the student's response is inappropriate. Learning from one's mistakes should be encouraged. Finally, this series of questions and answers reveals the answer to the problem. It is simpler and better to have a tutor formulate the series of questions. However, with some guidance, the student can often learn how to frame the questions himself. This is the goal we are striving for: self-confidence in solving word problems. Remember, there is often more than one way to solve a problem. So, if the solution follows the rules of algebra, do not discourage experimentation with methods other than the ones shown for these examples in this book.

1.2 Understanding - Not Memorizing

The main barrier to solving word problems is wanting to take what seems to the student to be the fastest and easiest approach: memorizing formulas and rules. **Understanding** how these formulas and rules arise and understanding definitions are keys to solving problems.

The following examples illustrate the principles and techniques involved. Although the examples relate to first year algebra problems, the same methods can be applied to word problems in any field of study.

1.3 Abstract Thinking

One requisite to learning algebra is the ability to think abstractly. That is, to use symbols to represent numbers. Although this may seem a simple concept, children need a degree of maturity to make this connection. Thus, if a child has great difficulty with the basic properties of numbers, arithmetic operations and equations expressed in symbols, he or she may not yet understand abstractions. If so, it may be wise to delay studying algebra.

1.4 Solid Foundation

Each step in mathematics learning builds on what came in previous steps. Because of this, I have found that the **easiest** way to learn mathematics is to score 100% on every test and "A" on every report card. Sometimes, a student will fall short of 100% on a test. It is vital to understand how to correct whatever caused the error in order to build a solid foundation for what follows.

Even if you think your answers are correct, checking them contributes to understanding the problem and the meaning of expressions used to solve the problem. Diagrams illustrating spatial relationships contribute to problem understanding and solution. They should be drawn whenever appropriate. Finally, tables often help to organize aspects of a problem that lead to an equation or equations that can be used to solve the problem.

1.5 General Problem Solving Guidelines

- Carefully read the problem. Don't just skim over it like reading a novel. You may need to reread it to be sure you **understand** what is given and what is to be found.
- Have you seen a similar problem that you have solved before?
- Look at the question at the end of the problem. It may help you choose an unknown or unknowns. Usually, try to choose as few as necessary to simply express all of the unknown quantities. Assign symbols to the unknowns. Write down what each symbol stands for. Use symbols that relate to the unknowns - not just X, Y and Z. For example, you might use "L" to represent length, not "X."
- Form expressions for all problem quantities in terms of the symbols you choose for the basic unknown or unknowns. Write what these expressions represent. For example, if the problem states that a board is 8 feet long and an unknown length is to be cut-off, the remainder would be $8 - L$.
- Look for statements that equate functions of the various unknowns and expressions. The number of equations needed equals the number of basic unknowns. For example, if the problem states that the remainder of a board 8 feet long is five times the length cut off: $8 - L = 5L$.
- Once you have solved the problem, try to check your answer. If you can't check, at least see if the answer seems reasonable using common sense. For example, for the above equation, add L to both sides: $8 = 6L$. Divide by 6 to get $L = 1\,{}^{1}/_{3}$. So, the remaining length will be $6\,{}^{2}/_{3}$ which is 5 times the length cut-off. The answer checks.

1.6 Solution of Algebraic Equations

Once a student understands the word problem and derives the appropriate equations, he must still find the solution. I assume that the student will be taught how to solve algebraic equations in class, but will have trouble generating the equations from a word problem statement. However, I have included descriptions of various equation solution methods in all of the following chapters, so the student can understand how these methods arose. This understanding will make him or her more confident when applying them. Also, formulas and methods can be derived, if forgotten.

1.7 Coordination with Algebra Class in School

A standard algebra text will provide the principles of algebra and the many problems needed to master algebra. This booklet will aid a tutor in helping the student to understand these principles and not be intimidated by word problems. As each problem is solved, it will provide a sense of accomplishment that encourages the student to continue.

To the student: Do not just read the solutions in this book! First, try to work each problem yourself without help. If you are stumped and are working without a tutor, uncover the first of this book's questions following the problem statement. After you have your own answer, see how your answer compares with the book's answer. Continue this process as long as you need help finding the problem solution. Also, if your school has a math club, join it. Your problem solving skills will benefit and you might find it to be fun!

2. Word Problems That Require Only Arithmetic

Some word problems are simply arithmetic problems expressed in sentences. For example:

If you travel at 30 miles per hour for 3 hours, how far do you go?

These problems do not usually cause any trouble, if the student knows the relation distance = speed x time. A somewhat more abstract version of this problem would be to substitute "s" for 30 miles per hour and "t" for 3 hours. Again, this transition will be easy for students who can think in abstractions.

2.1 Problems Requiring Comparisons

One type of problem, though exceptionally simple, causes many students a great deal of trouble even though it involves only arithmetic - no algebra. These problems involve comparison of prices per unit of various commodities. For example:

Tom had to decide which was the better deal, 12 ounces of cereal for $1.62 or the larger, 20 ounce size of the same cereal for $3.10.

Laziness may cause students to jump to the wrong conclusion and not do the calculations for this type of problem because of their experience or intuition. These questions could be asked:

What does Tom need to compare?

The cost of cereal per ounce (sometimes students will calculate ounces per dollar, which is usually harder; however, they forget to choose the larger ratio as best).

How much does each quantity of cereal cost per ounce?

The 12 ounce size costs $\$1.62/12 = \$0.132$ per ounce. The 20 ounce size costs $\$3.10/20 = \$0.155$ per ounce.

Which size is the better value?

The one with the lowest cost per ounce: the 12 ounce size.

Oddly enough, this situation, the smaller package being the better choice, is not unusual. Particularly, when sales or discount coupons are involved.

2.2 Solution by Ordered Elimination

Some problems require doing only arithmetic, but doing the operations in a certain **order**. For example:

A farmer had only pigs and chickens. His daughter went to school and made up a riddle for the other children to solve: "We have pigs and chickens that have a total of 40 legs and 16 wings. How many pigs and how many chickens do we have?

Solve this problem by a process of elimination. Pigs and chickens both have legs, but not wings. So, we can find the number of chickens by dividing the number of wings by two. Thus, there are 8 chickens. By a process of elimination, there are 40 - 2 x 8 = 24 pig's legs. By dividing 24 by 4 we find that there are 6 pigs.

So what questions should enter a student's mind when he or she sees this type of problem?

Since I am told only totals, how can I find specific quantities?

Maybe there is enough information to find one quantity. Yes, there is!

By using this quantity, can I simplify the problem to make it easier to find the remaining quantities? Yes.

2.3 Successive Application of Fractions

Consider the following problem:

A stock drops by 20%. How much must the stock rise to recover to its initial price?

On seeing this problem, many students will suspect a trick. Not knowing how to attack the problem, they will guess that the trick is that the obvious solution, 20% is correct. Unfortunately, the guess is wrong! Step by step, let's develop the correct solution.

What fraction of the initial stock price remains after a 20% drop?

80%. Or that is 0.8.

What does 0.8 have to be multiplied by to get back to 1 (100%)?

0.8 x X = 1, divide by 0.8 to remove the X multiplier: X = 1/(0.8) = 1.25.

What is 1.25 in percentage?

1.25 x 100% = 125%

So, how much must the stock price increase?

25%

Averaging consecutive fractional changes is tricky. On the surface it seems that we could average -20% and +20% to get no change. The only situation where this is valid is when both percentages are applied to the same initial amount. This would be the case if we were applying the changes to two different stocks with the same initial price. In the problem above, the 20% drop lowers the amount that is applied to the subsequent increase, so it must be greater than 20% to recover the initial value. For example, for a stock costing $100 would be worth only $80 after a 20% drop in price. From this level a 20% gain wold be $16 to only $96. To reach $100 from $80 would require a gain of $20 which wold be 25%.

2.4 Averaging Rates

In the same way and for much the same reason, averaging rates violates a student's "common sense" like successive application of fractions. A rate is a quotient. That is, one number divided by another. For example, speed is distance/time. The only situation where time rates can be averaged is when they have the same denominator. In the case of speed, this means that all rates must have the same duration. When in doubt, use the definition of rate to find the average. For speed, use:

average speed = total distance/total elapsed time [1]

The following problem illustrates a situation where averaging and using equation [1] produce the same result.

A car travels for 3 hours at 30 miles per hour. It then speeds up to 50 miles per hour for another 3 hours. What is its average speed?

If we average in the usual way: that is, by adding the quantities and dividing by the number of quantities, we get (30 + 50)/2 = 40 miles per hour. Now use equation [1]. Answer the following questions:

How can we find the numerator of [1], the total distance?

Add the distance during each part of the trip.

How can we find the distance traveled?

Distance = speed x time.

So, how far did the car go?

Total distance = 30 x 3 + 50 x 3 = 90 + 150 = 240 miles.

What was the total time?

Total time = 3 + 3 = 6 hours.

Use equation [1]. What is the average speed?

Average speed = 240/6 = 40 miles per hour.

As explained before, both simple averaging and equation [1] give the same answer.

Now, we will solve some problems where simple averaging and equation [1] give different results.

A car goes 10 miles at 30 miles per hour. It then speeds up to 50 miles per hour and goes an additional 10 miles. What was the average speed of the car?

A simple average of the speeds (33 + 50)/2 still gives 40 miles per hour. Let us calculate the average speed using equation [1].

What is the total distance travelled?

10 + 10 = 20 miles.

How long does it take for the first part of the trip?

Since speed x time = distance, time = distance/speed, [2]

time = 10/30 = 1/3 hour.

How long does it take for the second part of the trip?

Time = 10/50 = 1/5 hour.

So, what is the total time?

1/3 + 1/5 = (5 + 3)/15 = 8/15 hour.

What is the average speed, using equation [1]?

Average speed = 20/(8/15) = 300/8 = 37 1/2 miles per hour.

What does this mean? By solving [1] for total distance, we get:

Total distance = average speed x total time

That is, if we make the trip in the same amount of time, we will go the same total distance. This is the true meaning of average speed. Going at 40 miles per hour for 8/15ths of an hour would take us more than 20 miles.

As an extreme example of the discrepancy of the two methods of finding the average speed, consider:

A car goes 30 miles an hour for 10 miles. How fast must it go for the next 10 miles to average 60 miles per hour?

A simple average, (30 + S)/2 = 60, would suggest that going 90 miles an hour during the second 10 miles would be sufficient. Let's see what equation [1] tells us.

What is the total distance?

10 + 10 = 20 miles.

How long does it take to travel the first part of the trip?

Using equation [2]: time = distance/speed = 10/30 = 1/3 hour [3]

How long do we have to make the entire trip?

Using equation [1] and solving for total time gives:

total time = total distance/average speed = 20/60 = 1/3 hour [4]

Notice that [3] and [4] are equal. Note that in order to average 60 miles per hour over a 20 mile distance, you would need to complete the trip in 1/3 of an hour. However you already spent 1/3 of an hour while going 30 miles per hour during the first part of the trip all the available time for the entire trip was used in going the first 10 miles. So no matter how fast you drive the second 10 miles, it is impossible to average 60 miles per hour!

3. Simple Word Problems in One Variable

3.1 Solution of Algebraic Equations

Before considering word problems, I will describe principles that I have found that help beginning algebra students solve simple equations in one variable. These principles are studied in class, but expressed in a different way here.

- Clearly state what each symbol represents, particularly X.
- The goal is to isolate the variable on one side of the equation.
  - Whatever you do to one side of an equation, do to the other side of the equation.
  - Isolate the variable, simplify the equation, step by step. Do this by "undoing" various operations that are part of the equation. "Undoing" involves doing the "opposite" operation. For example:

    subtraction "undoes" addition;
    addition "undoes" subtraction;
    division "undoes" multiplication;
    multiplication "undoes" division.

Students must be careful when multiplying or dividing to include all terms oneach side of the equation.

- Whenever possible, combine like terms. By like terms we mean constants or terms containing the same variable to the same power. For example: we can combine 25 and -3; -2X and 3X/5; $3X^2$ and $-X^2$. To combine like terms, you may need to remove parentheses. Remove the innermost ones before the outer ones. You may also need to multiply expressions such as (aX + b)(cX + d),

  which results in $(ax)(cX) + (aX)d + b(cX) + bd = acX^2 + (ad + bc)X + bd$.

  For example, $X^2 - [(X - 3)(X + 7) - 25] = X^2 - [X^2 + 4X - 21 - 25] = X^2 - X^2 - 4X + 21 + 25 = - 4X + 46$.

- For word problems, make sure that when you solve the equation, the variable chosen is the solution to the problem. It may be necessary to use the variable in a simple expression to solve the problem.

As an illustration of algebraic equation solution, consider:

.2X - 12 = X/3 + 6

It is usually best to get rid of fractions (division) as the first simplification step. Using the "undo" principle, multiply to remove divisions.

Multiply by 5: X - 60 = (5/3)X + 30

Multiply by 3: 3X - 180 = 5X + 90

Move constants to the right-hand side of the equation by adding 180 in order to remove -180: 3X = 5X + 270

Isolate the variable by moving all terms containing X to the left-hand side. Eliminate 5X from the right hand side by subtracting 5X from both sides:

3X - 5X = 270

Combine like terms: -2X = 270

Divide by -2 to eliminate the multiplication and completely isolate X: X = -135.

(X could be isolated on the right-hand side. This would be slightly simpler, but might confuse the beginning student.)

These methods will be illustrated often when solving the equations generated by the word problems that follow.

3.2 Number Problems

Number problems involve relationships between numbers. Usually the numbers are integers and often positive integers.

Eight less than three times a number is twenty-five. What is the number?

What symbol do you want to assign to the unknown number?

Let's use "N". (Any symbol could be used.)

We need to have something to take eight away from. What should it be?

It must be "three times a number". So, 3N - 8.

What does the problem tell us this equals?

Twenty-five. So, 3N - 8 = 25.

Can you "undo" the subtraction of 8 and the multiplication by 3?

Add 8 to both sides: $3N = 33$. Divide by 3: $N = 11$.

Does your answer check?

$3x11 - 8 = 25$, $33 - 8 = 25$, $25 = 25$. Yes.

A judicious choice of variables is often important. A beginning student may select as a variable the first unknown mentioned in the problem. Usually, the best choice is the one that defines the other unknowns. The following problem illustrates this choice.

Three numbers are such that the first is three times the second minus 2, and the third is 2 more than twice the first. The sum of the three numbers is 106. What are the three numbers?

A typical dialog between the tutor and student could be:

Is there an unknown that is not defined as a function of the others (it stands alone)?

Yes, the second number.

Let us use that number as our variable, what shall we call it?

How about "s" for second? (Any symbol could be used.)

Is there an unknown that is a function of "s"?

Yes, the first is: $3s - 2$. [1]

Is there another unknown that is a function of the first or second unknown?

Yes, the third is: $2(\text{first}) + 2 = 2(3s - 2) + 2 = 6s - 4 + 2 = 6s - 2$. [2]

The problem tells us what the sum is. What is an equation for it?

$(3s - 2) + (s) + (6s - 2) = 106$.

Can you solve this equation?

Combining like terms gives: $10s - 4 = 106$; $10s = 110$; $s = 11$. [3]

Substitute [3] into [1] and [2] to find the first and third numbers. What are they?

First number: $3(11) - 2 = 31$. Third number: $6(11) - 2 = 64$.

Does the answer check?

$31 + 11 + 64 = 106$. Yes.

When an unknown quantity appears more than once in the problem statement, it can confuse the beginning student. Also, if a fraction of the unknown appears, that can add to the confusion. For example:

The difference of two numbers is 20. If the larger is divided by the smaller, the quotient is 2 and the remainder is 7. What are the two numbers?

Since there is a simple relationship between the two numbers, "difference of two numbers is 20". We only need one variable. What symbol do you want to use?

Let's use F for the first number. Then the second number would be F + 20 (we could use F - 20, the result will be the same). So the smaller is F and the larger is F + 20.

To understand how to work this problem, you need to understand how to write a number as the sum of a quotient and a remaining fraction. For example:

Can you write 13 divided by 5 as the sum of a quotient and a remainder?

$13/5 = 2 + 3/5$

How can the equation defined by the problem be expressed?

$(F + 20)/F = 2 + 7/F$

What should be done first to simplify this equation?

Remove fractions. "Undo" divisions by multiplying by F: $F + 20 = 2F + 7$.

Solve the problem. Does your answer check?

$-F = -13$, $F = 13$, $F + 20 = 33$, the numbers are 13 and 33.

Check: $33/13 = 2 + 7/13$. Yes.

Some problems will involve separating a number (for example 31) into two parts. If one of the parts is symbolized by P, the other becomes 31 - P. Hence, when the parts are joined (added) the result is $P + (31 - P) = 31$.

3.3 Motion Problems

Some motion problems are very simple. One thing that can cause trouble is having differing measurement units involved. For example, if times are in minutes and speeds are in mph (miles per hour). Units should be consistent either by converting miles per hour to miles per minute, or more simply, by converting minutes to hours. The beginning and ending clock times should not be used directly. Instead, use time intervals until the algebraic equations are solved. Also, some students have trouble deciding whether to add or subtract speeds and distances. For example:

Two friends are driving cars on a freeway. One starts from town A, going 50 miles per hour toward town B, that is 330 miles away. The other starts from town, B, going 60 miles per hour toward town A How much time will elapse before they meet? Where will they meet?

Can you draw a diagram that illustrates the problem? This should be done whenever possible.

A ⟶ **330 Miles** ⟵ B

50 mph **60 mph**

What is the relation between distance, speed and time?

Speed x time = distance.

We know speeds and total distance. What don't we know to use the formula above?

We don't know the time. Let's call it T.

Can you write an equation for the total distance?

First car distance + second car distance = total distance. $50T + 60T = 330$.

[Note: Add the two distances, since each car contributed to covering the distance. Occasionally a student will subtract the distances.]

How much time elapsed before they met?

$110T = 330$, $T = 3$ hours.

Where will they meet?

$50 \times 3 = 150$ miles from town A.

Some travel problems involve going upstream or downstream on a river. Or, going upwind or downwind in an airplane. For example:

Jane can go 20 miles/hour in her boat in still water. Today there is a current flowing.

She travelled 5 miles upstream in the same time it took her to go 7 miles downstream. How tast is the current?

What do you want to call the unknown?

Let's call it C for current speed.

How fast does the boat go upstream and downstream?

Upstream: 20 - C. Downstream: 20 + C.

What quantities are equal for this problem?

The times going upstream and downstream.

Knowing distance and speed, how can the time be found?

Since, speed x time = distance, time = distance/speed.

What are upstream and downstream times?

Upstream: 5/(20 - C) Downstream: 7/(20 + C)

These times are equal. Can you solve the resulting equation?

5/(20 - C) = 7/(20 + C). Remove divisions by multiplying by the common denominator, (20 - C)(20 + C): 5(20 + C) = 7(20 - C). Remove parenthesis: 100 + 5C = 140 - 7C. Collect like terms: 12C = 40. Divide by 12: C = 3 1/3 miles/hour.

Does your answer check?

Time to go upstream = distance/speed = 5 /(20 - 10/3) = 5/(50/3) = 3/10. Time to go downstream = 7/(20 + 10/3) = 7/(70/3) = 3/10. So, the times are equal.

3.4 Coin Problems

These problems are usually fairly simple when values are expressed in cents. The same methods apply to bills, stamps and anything that comes in denominations.

John has only nickels and quarters in his pocket. He has 15 coins that add up to $2.35. How many nickels and quarters does he have?

What do you want to use to symbolize the number of nickels or the number of quarters?

Let's use N for the number of nickels.

How can the number of quarters be expressed?

15 - N (this is a very common tactic - subtracting the unknown amount of one variable from the total for both unknowns to represent a second unknown).

What are nickels and quarters worth?

Nickels are worth: 5N. Quarters are worth: 25(number of quarters) = 25(15 - N).

Can you write an equation that relates the value of the nickels and quarters to the total John has?

Value of nickels + value of quarters = total value, 5N + 25(15 - N) = 235.

What is the value of N?

Remove parenthesis: 5N + 375 - 25N = 235. Collect terms: -20N = -140. Divide by -20 to "undo" the multiplier of N: N = 7.

How many quarters were there?

Number of quarters = 15 - N = 15 - 7, so, there are 8 quarters.

Do the numbers ot nickels and quarters seem correct?

Yes, since 5x7 + 25x8 = 35 + 200 = 235.

Now we are ready for a more complicated coin problem. For this problem a table will help organize the information so the solution is easier to formulate.

Jane has a coin purse that contains only pennies, nickels and dimes. In all, she has $1.55. There are three times as many pennies as dimes. There are also twice as many dimes as quarters. How many coins of each type are in the purse?

Is there some unknown quantity that defines the other quantities?

The number of dimes. Let's call it D for dimes.

A table can be constructed by the student to help organize the solution of word problems. Think of the problem as if it were a play. In this problem, the "actors" are coins: pennies, nickels and dimes. List these "actors" on the left as row labels. The column headings are different "characteristics" of the roles the "actors" play. That is, what we know or can deduce about the "actors" roles. These characteristics can be: speed, time interval, distance; percent, volume or weight, amount; value of each, number, total value; etc. The table should be constructed so the columns lead up to one or more final columns that can be used to set up the word problem equations. Row entries in these last columns combine to form equations. Not all table spaces need to be filled in some tables. Some "characteristics" are unknown or not needed.

Construct a table with as much information as you can obtain from the problem statement. What are the "actors"?

The coins: nickels, dimes and quarters. Hint: do not forget "all coins."

What "characteristics" of the "actors" are important for this problem?

The number of coins of each denomination, the value of each coin and the total values.

Coin	Number	Value ot each	Total value
Penny	3D	1	3D
Nickel	D/2	5	5(D/2)
Dime	D	10	10D
All coins	3D + D/2 + D	-	155

Note that all table places have meaningful or needed data. In this table, the number of all coins is known but not needed to solve this problem. The space for "Value of each" for "All coins" is meaningless. Many students forget to add a last row that includes information from the preceding rows, usually as totals.

Having set up the table, can you write an equation for the total value of the coins?

From the last column: 3D + 5D/2 + 10D = 155.

Can you solve this equation and find how many coins ot each type there are?

Clear traction by multiplying by 2: 6D +5D +20D = 310. Collect terms: 31D = 310. Divide by 31 to "undo" the tactor of 31: D = 10. Number of pennies = 3D = 30. Number of nickels = D/2 = 5.

Thus there are 30 pennies, 5 nickels and 10 dimes.

Does your answer check out?

$30 + 5x5 + 10x10 = 30 + 25 + 100 = 155$. Yes.

3.5 Digit Problems

A digit problem involves the individual digits that make up a number. The following is a simple example:

The unit's digit of a two digit number is 1 greater than twice the ten's digit. It is also 5 more than the ten's digit.

Is there an element in the problem that is the basis for a simple description of the other element?

The ten's digit defines the unit's digit in the last sentence of the problem. So, let T be the ten's digit. Then, unit's digit $= T + 5$.

From the first sentence of the problem, what is another expression for the unit's digit?

Unit's digit $= 1 + 2T$.

What equation will allow you to find T?

Equate the two expressions for the unit's digit: $T + 5 = 1 + 2T$.

Collect like terms: $T = 4$.

What is the unit's digit? What is the number?

Unit's digit $= T + 5 = 4 + 5 = 9$. Number is 49.

The following problem involves the two digit number itself. Otherwise it is no more complicated than the previous problem.

The unit's digit of a two digit number is 2 more than the ten's digit. The sum of the digits is a quarter of the two digit number.

If you know the ten's digit and the unit's digit, what is the number?

Number = 10 x ten's digit + unit's digit. [1]

What do you want to use for the unknown? What do you want for its symbol?

The unit's digit is a function of the ten's digit. So, let T be the ten's digit.

What is the unit's digit and the number a function of T?

Unit's digit $= T + 2$. From equation [1]: Number $= 10T + (T + 2) = 11T + 2$.

Can you write an equation based on the last sentence of the problem statement?

$T + (T + 2) = (11T + 2)/4$. Collect terms on the left: $2T + 2 = (11T + 2)/4$.

What is the ten's digit? What is the unit's digit?

"Undo" the division by multiplying by 4: $8T + 8 = 11T + 2$. Collect like terms:

$3T = 6$, $T = 2$. Unit's digit $= T + 2 = 4$

What is the number? Does it satisfy the problem conditions?

Number is 24. $2 + 4 = 24/4$, $6 = 6$, Yes.

There are problems where the unknown must appear on both sides of the equation. The following problem illustrates this situation:

A 3 digit number has the ten's digit 3 greater than the unit's digit and the hundred's digit 2 less than the ten's digit. The sum of the tens and hundred's digits is four times the unit's digit.

What is a convenient variable for this problem?

The unit's digit is referred to several times. Specifically in the "equation" sentence (last sentence that uses the word "is." "Is" can be used as an abbreviated form of "is equal to.") Call the unit's digit "U".

Using this variable, what is the ten's digit?

Ten's digit $= U + 3$.

What is the hundred's digit?

Hundred's digit = ten's digit $- 2 = (U + 3) - 2 = U + 1$.

What is the sum of the digits?

Sum $= 4U$.

Can you form an equation using the last sentence of the problem?

Ten's digit + hundred's digit = Sum; $(U + 3) + (U + 1) = 4U$; Combine like terms:

$2U + 4 = 4U$; $2U = 4$.

What do you get when you solve this equation for U?

"Undo" the multiplication by dividing by 2: $U = 2$.

What is the number?

Ten's digit $= U + 3 = 2 + 3 = 5$. Hundred's digit $= U + 1 = 2 + 1 = 3$. Number is 352.

Is the answer correct?

The ten's digit, 5, is three more than the unit's digit, 2. The hundred's digit, 3, is two less than the ten's digit, 5. The sum of the digits is $3 + 5 + 2 = 10$, which is five times the unit's digit, 2. So, the answer is correct.

A more complicated digit problem is the following:

The ten's digit of a two digit number is two more than the unit's digit. When the digits are reversed, the new number is three more than one-half the original number.

To work this problem, you need to think of how a two digit number can be written as a sum. For example:

How can a two digit number such as 29 be written as a sum?

$29 = 20 + 9$ or $2x10 + 9$.

What should be the variable for this problem?

Since the ten's digit is a function of the unit's digit, use the unit's digit, U, as the primary variable.

What function represents the original number?

Since the ten's digit is $U + 2$, we can write:

$10(U + 2) + U = 10U + 20 + U = 11U + 20$.

What would be the number if the digits were reversed?

$10U + (U + 2) = 11U + 2$.

Can you formulate an equation relating the two digit number and the reversed number?

Reversed number = (original number)/2 + 3. (Note:because it is close to the original word problem statement but still an equation, it is a good idea to formulate the equation this way using word descriptions of the numbers before writing a

symbolic equation). Thus, $(11U + 2) = (11U + 20)/2 + 3$.
How would you solve this equation? What should be the first step?
The first step should be to remove the division by 2 by "undoing" by multiplying by 2:
$2(11U + 2) = (11U + 20) + 6$. Remove parenthesis and collect terms:
$22U + 4 = 11U + 20 + 6$, $11U = 22$, $U = 2$, $U + 2 = 4$. So the number is 42.
Check your answer. Is it correct?
$24 = 42/2 + 3 = 21 + 3 = 24$. Yes.

3.6 Age Problems

Simple age problem:
Sean is 8 years old. His mother Helen is 27. How many years until Sean is half as old as is mother?
What do you want to call the unknown?
Call it Y for the number of years.
How old will Sean and his mother be in Y years?
Sean will be $8 + Y$. Sean's mother will be $27 + Y$.
What relates these two ages?
The last sentence of the problem: $8 + Y = (27 + Y)/2$
Can you solve this equation?
Get rid of the divisor by multiplying by 2: $16 + 2Y = 27 + Y$. Collect like terms: $Y = 11$ years.
Does this answer check out?
In 11 years, Sean will be 19 and his mother will be 38. 19 is one-half 38, so it checks.

Age problems are not particularly difficult if the student carefully follows the problem statement and organizes his or her work. A simple table is usually helpful.
Five years ago Bob was three times as old as Tommy. In two years from now Bob will be only twice as old as Tommy. How old are Bob and Tommy now?
What would be a good choice of variable?
Bob's age is stated in terms of Tommy's age. So, let T be Tommy's age now.
Create a table that shows what is known about the ages. Who are the "actors"?
Bob and Tommy.
What are the actors' characteristics?
Their ages five years ago, now and two years from now.

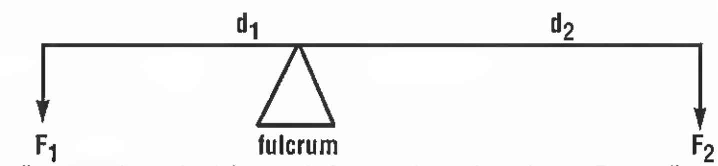
Person	Age now	Age five years ago	Age two years from now
Tommy	T	T - 5	T + 2
Bob	?	3(T - 5)	2(T + 2)

Now, using the last 2 columns of the last row of the table, what are two expressions for Bob's age now as a function of Tom's age now, T ?

From Bob's age 5 years ago add 5 years to get his age now: $3(T - 5) + 5$. From Bob's age in 2 years subtract 2 years to get his age now: $2(T + 2) - 2$.
Since these are two expressions for the same thing (Bob's age now), can you form an equation and solve it?
$3(T - 5) + 5 = 2(T + 2) - 2$. Remove parenthesis: $3T - 15 + 5 = 2T + 4 - 2$.
Collect terms: $3T - 10 = 2T + 2$, $T = 12$, so, Tommy is 12 years old now.
Use either expression for Bob's age now. How old is he?
$3(T - 5) + 5 = 3(12 - 5) + 5 = 3(7) + 5 = 21 + 5 = 26$. So Bob is 26 years old now.

3.7 Balance problems

The balance problems we solve involve the simple physics equation that relates to teeter-totters (see-saws) and levers. Consider the geometry of a teeter-totter:



d_1 and d_2 are the distances from the fulcrum (balance point) where forces F_1 and F_2 act.
The forces on a teeter-totter are the weights of the children on each side. From physics,
$F_1 \times d_1 = F_2 \times d_2$ [1] when the teeter-totter is balanced. From the diagram above, F_1 must be larger than F_2 since d_1 is shorter than d_2. For example, F_1 could be 10 and F_2 could be 5 when d_1 is 8 and d_2 is 16, since $10 \times 8 = 5 \times 16$. The units of measurement do not matter as long as the units of force are the same and the units of distance are the same. If one distance is in feet and the other is in inches, one must be converted to the other so they are the same. If there is more than one force acting on either side, the sum of force times distance from the fulcrum on one side must equal the sum of force times distance on the other side. Now solve a simple balance problem:
Sue's mother weighs 160 pounds. She is sitting 2 feet from the fulcrum of a teeter-totter. Sue weighs 40 pounds. How far from the fulcrum should Sue sit to balance her mother on the other side?
What symbol do you want to use for the unknown?
Let's use S for Sue's distance from the fulcrum. (Any symbol can be used.)
Make a table to relate all the quantities. Who are the "actors"?
Sue and her mother.
What do we know or want to find out about them?
Their weights and distances from the fulcrum.
From equation [1], what has to equal (balance)? (Add this quantity to your table.)
The products of force (weight) and distance for each person.

Person	Weight	Distance	Weight x Distance
Sue	40	S	40S
Sue's mother	160	2	320

Using equation [1], can you set up an equation for this problem?
From the last column: $40S = 320$.
How would solve this equation for S?
"Undo" the multiplier by dividing by 40: $S = 8$.
Does this answer check?
$40 \times 8 = 160 \times 2$. Yes.
The following problem involves more than two forces:
Three girls were playing on a ten foot teeter-totter with the fulcrum at the center. **Jane weighs 80 pounds and sat on one end. Sarah weighs 60 pounds and sat on the other end. Nancy weighs 25 pounds and sat on Sarah's side. How far from the fulcrum did Nancy sit to balance the teeter-totter?**
What symbol do you want to give to Nancy's distance from the fulcrum?
Let's call it N.
How far are Jane and Sarah from the fulcrum?
They are both 5 feet from the center on opposite sides.
Can you construct a table giving the "actors" and their "characteristics"?

Person	Weight	Distance	Weight x Distance
Jane	80	5	400
Sarah	60	5	300
Nancy	25	N	25N

Can you write a "balance" equation for this problem?
$80x5 = 60x5 + 25N$, $400 = 300 + 25N$, combine like terms: $25N = 100$.
So, how far is Nancy from the fulcrum?
After dividing by 25, we find that $N = 4$, so, she is 4 feet from the fulcrum $N = 4$.
Does your answer check?
$80 \times 5 = 60 \times 5 + 25 \times 4$, $400 = 300 + 100$, $400 = 400$. Yes.
The position of the fulcrum to achieve balance can also be found as the following problem shows.
Carl weighs 70 pounds and Buddy weighs 30 pounds. They found a ten foot board in their backyard. When they sit on each end with a rock acting as a fulcrum, where must the rock be placed to balance (ignore the weight of the board)?
What do you want to be your unknown?
Let the distance from Carl be C.
How far is the rock from Buddy?
It is 10 - C feet away.
Can you construct a table to help solve this problem?

Person	Weight	Distance	Weight x Distance
Carl	70	C	70C
Buddy	30	10 - C	30(10 - C)

What is the appropriate balance equation for this problem?
From the last column of the table: $70C = 30(10 - C)$
What is the solution of this equation?
Remove the parenthesis: $70C = 300 - 30C$, collect like terms: $100C = 300$, divide by 100 to "undo" the multiplying factor: $C = 3$ feet from Carl.
Does your answer check?
$3 \times 70 = 30\ (10-3) = 30 \times 7$. Yes.

3.8 Mixture Problems

Mixture problems often involve balancing the amount of a substance in each component with the amount in the resulting mixture. To illustrate, solve this simple problem:

A nurse wants to make a solution that is 5% salt by adding some volume of a 10% solution to 25 liters of a 2% solution. How much should be added?
What do you want to call the unknown volume of 10% solution?
Let us call it V for volume.
Make a table showing what is known and unknown. Who are the "actors"?
The different salt solutions.
What are the characteristics of the salt solutions? (Hint: we expect to need to find how much salt is in each solution: amount = fraction x volume.)
Their volumes, the fraction of salt they contain and the amount of salt they contain.

Solution	Volume	Fraction salt	Amount of salt
2%	25	.02	.5
10%	V	.10	.1V
5%	V + 25	.05	.05(V + 25)

What does the "Amount of salt" column tell you?
If I add the amount of salt in the 2% and 10% solutions, I should get the amount of salt in the 5% solution. So, $.5 + .1V = .05(V + 25)$
How would you solve this equation?
Eliminate fractions (decimal fractions) by multiplying by 100: $50 + 10V = 5(V + 25)$.
Remove the parenthesis: $50 + 10V = 5V + 125$. Collect terms: $5V = 75$.
Divide by 5 to "undo" the factor of 5. $V = 15$.
Can you check the result?
The amount of salt balances: $.02x25 + .10x15 = .05(15 + 25)$, $.5 + 1.5 = .05x40$, $2 = 2$.

Mixture problem involving dilution.
How much water must be added to 7 quarts of a 5% sugar solution to make a 3% solution?
What variable do you want to use for the unknown?
Let's use W for water volume. (As usual, any symbol could be used)
How do you want to organize the problem information? Who are the "actors"?
Water and the sugar solutions.
What are the "characteristics" of these "actors"?
Their volumes, the fraction of sugar and the amount of sugar each has.

Liquid	Volume	Fraction of sugar	Amount of sugar
Water	W	0	0
5% solution	7	.05	.35
3% solution	W + 7	.03	.03(W + 7)

What does the "Amount of sugar" column tell you?
If I add the amount of sugar in the water and in the 5% solutions, I should get the amount of sugar in the 3% solution. So, 0 + .35 = .03(W + 7)
How will you solve this equation?
Eliminate fractions (decimal fractions) by multiplying by 100: 35 = 3(W + 7).
Remove the parenthesis: 35 = 3W + 21. Collect terms: 3W = 14.
Divide by 3 to "undo" the factor of 3: W = 4 2/3 quarts.
Can you check the result?
See if the amount of sugar balances: .05x7 = .03x(4 2/3 + 7),
.35 = .03x(14/3 + 7), .35 = .03x35/3 = .35. Yes, it does balance.

3.9 Business Problems

I consider pricing, investment and tax problems as some categories that can be included as business problems. Consider the following investment problem:

Henry has a bank savings account that pays 2% interest. He also has a certificate of deposit that pays 6%. His interest income is $250 on a total investment of $5000. How much money is in each of his investments?

What do you want to choose as a variable?
It does not matter, but you may want to choose S for savings account since it is the initial letter of the variable.
Can you set up a table showing the information contained in the problem? Who are the "actors"?
The investments and the total invested.
What are the "characteristics" of these investments?
The amounts invested, their yields and the interest earned.

Investment	Amount	% yield	Interest
Bank Savings Account	S	2	.02S
Certificate of Deposit	5000 - S	6	.06(5000 - S)
Total Investment	5000		250

Does the last column suggest an equation to you?
Yes. Bank interest + Certificate of Deposit interest = total interest.
.02S + .06(5000 - S) = 250.
How would you solve this equation for S?
Remove decimal fractions by multiplying by 100: 2S + 6(5000 - S) = 25000. Remove parenthesis: 2S + 30000 - 6S = 25000. Collect terms: -4S = -5000. Divide by -4: S = 1250. So the bank account contains $1250. The Certificate of Deposit is for 5000 - 1250 = $3750.
Do your answers check?
.02 x 1250 + .06 x 3750 = 25 + 225 = 250. Yes.

A problem that concerns retail pricing follows:
Sam has a bicycle that he bought from a wholesaler for $90. He is concerned that if the bicycle does not sell in a few weeks, he may have to have a 10% off sale. How should he price it initially to assure a 20% profit over his cost even when it is sold at the sale price?
Assume the bicycle must go on sale. That way, we know what the profit percentage must be.
What is it we need to find? What symbol do you want to give it?
The initial price. Let's call it I.
Can you write an equation in words, that relates cost, sales price and profit?
Sale price - cost = profit. [1]
What is the sale price in terms of the initial price?
Sale price = .9 x initial price = .9I (Note: the .9 came from 100% - 10% = 90%) [2]
What can you substitute for "profit" in equation [1]?
Profit = .2 x cost = .2 x90 = 18 [3]
Using equations [2] and [3], what does equation [1] become?
.9I - 90 = 18, .9I = 108, Remove decimal by multiplying by 10: 9I = 1080
Divide by 9: I = 120. So, the initial price should be $ 120.
Can you check this answer?
10% sale produces a price of 120 - 12 = 108. Profit would be 108 - 90 = 18 that is 20% of the $90 cost.

As example of a somewhat complicated tax problem is the following:
Jenny had a job last year. She forgot how much she made, but knows how much was left over after taxes were withheld. She took home $34,000. She knew she paid

about one-quarter of her income in tax after deducting $19,000 standard deduction. What was her pretax income?

So, after the initial shock on reading this word problem, what questions can be asked and answered?

What do we want to find? What symbol shall represent it?

We want to find the pretax income. Let's call it, "P" (or "X", or any desired symbol).

(For this problem, we can use the "**top down**" approach.

That is, we start with the most general statement of the problem and refine it by replacing general items with more specific expressions.)

What is the most general statement we can make?

If we take the pre-tax income and subtract the tax, we get the take-home pay. Combining symbols and numerical data, gives: P - tax = 34000.

How can we find the tax?

The problem statement says, the tax is 1/4 of the income after deduction.

That is: tax = 1/4 x (income after deduction). So, our complete equation becomes: P - (income after deduction)/4 = 34000.

Finally, what is the "income after deduction"?

From the problem statement: Income after deduction = Pretax income - standard deduction. Or, symbolically: income after deduction = P - 19000.

The complete statement is: P - (P - 19000)/4 = 34000.

Now we have changed the word problem into an algebraic equation. It can be solved by methods the student should have already practiced:

Get rid of the fraction by multiplying each term on both sides of the equation by 4: 4P - (P - 19000) = 136000. (Often students will forget to multiply the first term, "P", by 4).

Remove parentheses: 4P - P + 19000 = 136000 (sometimes students forget to reverse the sign of "- 19000").

Combine like terms: 3P + 19000 = 136000.

Subtract 19000 from both sides to isolate the variable: 3P = 117000.

Divide by 3 to get the solution: P = 39000.

A related method is called "**bottom-up**". It starts with most basic detail and works up to the more involved. You may want to skip this description for now, if you think it will confuse the student. If not, for this problem:

What is the most basic detail?

The last part of the problem statement, "after deducting a $19,000 standard deduction.

How can that be expressed algebraically?

P - 19000. This is what remains to tax after subtracting the standard deduction.

What is the next level of complication?

Let's figure the tax.

What expression will result in the tax?

From the problem description, we can write: 1/4 times what remains after subtracting the standard deduction as: (P - 19000)/4.

Finally, we can write: pretax - tax = 34000. Or, symbolically, P - (P - 19000)/4 = 34000. This is the same equation that we arrived at using the "top-down" method.

Usually, the top-down method is better. Occasionally for very difficult problems the two methods can be combined. We will use the top-down method for several future problems.

3.10 Proofs of Number Properties

Another type of problem is more abstract, but still simple, if the student understands some general definitions. These problems involve proofs of odd and even number properties. One such problem would be:

Prove that when two odd numbers are added, an even number results.

The tutor or student can ask the following series of questions with corresponding responses:

What is an even number?

An even number is an integer that is divisible by 2.

How can I represent an even number symbolically? (Remember we need a general proof, not just specific examples).

A number multiplied by 2 is always divisible by 2. So, how about 2 x n (or more compactly, 2n, where n is an integer).

Could I use a symbol other than n?

Sure, there is nothing magical about n.

What is an odd number?

An odd number is 1 greater (or less) than some even number.

How can I represent an odd number?

Since an even number is 2n, an odd number would be 2n+1.

We need 2 odd numbers, how can we distinguish them?

Let's call the second one 2m+1, since we can substitute any letter for n.

How can we represent the sum of these two odd numbers?

(2n+1) + (2m+1)

Can I rewrite this so it looks like an even number?

2n+2m+2 = 2(n+m+1) = 2(integer). Since n, m and 1 are integers, n+m+1 is is an integer also.

Although this process is long for a simple proof, it illustrates the type of dialog a beginning algebra student needs to go through to **understand** abstract reasoning.

A simple word problem that uses the properties of integers is the following:

What three consecutive integers add up to 30 ?

We will go through a reasoning process similar to the one we used for solving the previous proof:

What symbol will we choose to represent some unknown integer?

How about "n", since it doesn't matter what we choose.

What is the next consecutive integer after n?

That would be n+1. The one after that would be (n+1)+1, or n+2.

How can we express our problem using these constructs?

(n)+(n+1)+(n+2) = 30; so, 3n+3 = 30; subtracting 3 from both sides, gives 3n = 27; finally, dividing by 3, gives n = 9. The answer is 9, 10 and 11.

Although this and other simple problems can be solved by trial and error, that method is unacceptable. The student should be practicing abstract reasoning. A common sense solution of dividing the sum, 30, by the number of integers, 3, to get the middle integer, shows ingenuity. However, it is not an algebraic method. It would be, it the student used "n" for the middle integer, then wrote:

(n-1) + n + (n+1) = 30; 3n = 30; n= 10.

3.11 Geometry Problems

Some problems seem incomplete to the student. These problems may depend on information learned in a previous course. For example, the geometric properties ot various figures:

Triangle: Sum of angles = 180.
Area = height x base/2
Right triangle: One 90 degree angle
Sum of the squares of the lengths of the legs = square of the length of the hypotenuse
(Pythagorean Theorem) (The hypotenuse is the side opposite the right angle)
Rectangle: Area = length x width
Angles = 90 degrees each. Sum of angles = 360 degrees.
Circle: Perimeter = 2 pi x radius = pi x diameter
Area = pi x radius^2
Central angle = 360 degrees

These problems also require the student to use consistent units. If both inches and feet are used, one should be converted to the other. As an example of a problem requiring the student to know a general property of triangles, consider:

One angle of a triangle is twice a second angle, the third is half again as large as the second angle. How large are the angles?

The following questions may help the student cope with this problem:

What is the first phase in solving word problems?

To generate an equation that can be solved by algebraic methods.

What seems missing from this problem statement?

Something that equals something else!

What equality do you remember about triangle angles?

The sum of the angles of a triangle is 180 degrees.

Now that we have a chance to write an equation, choose a variable.

The second angle is not expressed as a function of any other angle. Call the second angle, "S".

What will the first angle be?

2S

What will the last angle be?

3S/2 (Many students have trouble with "half again" - it means 1 1/2 times)

Now, can you write the equation ot the sum of angles?

2S + S + 3S/2 = 180.

How do you get rid of fractions?

"Undo" the division by multiplying by the common denominator: 4S + 2S + 3S = 360.

Combine like terms.

9S = 360; S = 40.

What are the other angles?

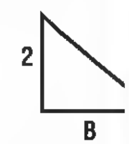
First: 2S = 2(40) = 80. Third: 3S/2 = 3(40)/2 = 60.

A property of right triangles is needed to solve the following problem:

A carpenter has 8 feet of lumber to make a right triangular truss. The height must be 24 inches. How wide should the base be and how long the diagonal (hypotenuse)?

Are units consistent?

No. Let's convert 24 inches into 2 feet.



What relationship connects the legs and hypotenuse of right triangles?

The Pythagorean Theorem: $leg_1^2 + leg_2^2 = hypotenuse^2$ [1]

What do you want as a variable?

Let's use the base. Call it B. [It doesn't matter much which side.] B

Since we know or assign the lengths of two triangle sides and the total length, what is the length of the third (diagonal or hypotenuse) side?

Length of third side = total length - lengths of other two sides.

Hypotenuse = 8 - 2 - B = 6 - B [2]

What does equation [1] become using the triangle sides for this problem?

$B^2 + 2^2 = (6 - B)^2$ [3]

How would you solve this equation?

Square the terms in [3]: $B^2 + 4 = 36 - 12B + B^2$. Collect like terms: 12B = 32.

Divide by 12: B = 32/12 = 8/3 = 2 $^2/_3$.

From [2]: Hyp = 6 - 8/3 = (18 - 8)/3 = 10/3 = 3 $^1/_3$.

Can you check your answers?

Total length = 2 + 8/3 + 10/3 = 2 + 18/3 = 2 + 6 = 8. Pythagorean Theorem:

$2^2 + (8/3)^2 = 4 + 64/9 = 36/9 + 64/9 = 100/9 = (10/3)^2$. Checks.

Occasionally a problem that seems to involve solving a quadratic equation (one involving the square of a variable) reduces to a linear problem because the squared terms cancel. The following problem shows how this can happen:

A rectangular tablecloth is 24 inches longer than it is wide. It its width were increased by 12 inches and its length decreased by 15 inches, it would have the same area as betore. What are its dimensions now?

What would be the most convenient variable to use in solving this problem?

The original width seems the most basic. Let's call it W.

What is the original length?

W + 24.

What would be the new length?

W + 12.

What would be the new length?

(W + 24) - 15 = W + 9.

What is the formula for the area ot a rectangle?

Area = width x length.

Can you construct a table that shows this information?

Tablecloth	Width	Length	Area
Original	W	W + 24	W(W + 24)
Modified	W + 12	W + 9	(W + 12)(W + 9)

Can you use the last column to write an equation appropriate for this problem?

W(W + 24) = (W + 12)(W + 9).

What do you get when the multiplications are performed?

$W^2 + 24W = W^2 + 21W + 108$.

Can you simplify this equation and solve for W?

Collect like terms: W2 - W2 + 24W - 21W = 108, 3W = 108, W = 36 inches.

What is the original length and area?

Length = W + 24 = 36 + 24 = 60 inches. Area = 36 x 60 = 2160 square inches.

What would be the modified width, length and area be?

Width = W + 12 = 36 + 12 = 48 inches. Length = W + 9 = 36 + 9 = 45 inches.
Area = 48 x 45 = 2160 square inches.

Does the problem answer check.

Yes, since both areas are the same.

4. Word Problems Using Two Linear Equations:

Sometimes we can take a problem that has two variables and replace one by a simple tunction ot the other. This transtorms the problem so that an equation in a single variable results. For example, the carrot patch problem involved length and width. Since the width could be expressed as 30 - length, we needed only to solve a problem involving the length. For other problems it may be simpler to set up two equations in two unknown variables.

When these equations are of the form (or can be transformed into the form): aX + bY = c , they are called **linear equations** because when they are graphed, the results are straight lines. Most beginning students have a tendency to introduce more variables than are necessary. When as many equations can be formulated as there are variables, the problem can be solved. This is true because each equation could allow one variable to be expressed in terms of some of the remaining variables. Using several variables and several equations is often more ditticult than using simple relationships with one variable to generate a single equation. But, there are many problems where the relationships between the unknowns are not particularly simple. The problems in this section are like that.

4.1 Mixture Problem (Combining Two Amounts)

To illustrate two variable linear problems, consider the following:

A chemist wants to make a compound, "C", that contains 50 grams of nickel and 40 grams ot chlorine. He has two chemicals to start with. The first, "A", is two-thirds nickel and one-third chlorine. The second, "B", is one-fourth nickel and three-fourths chlorine. How many grams of "A" and of "B" will be needed?

Since the problem statement ends with the goal of the problem solution, we can dispense with that question.

What names can we assign to the amounts of chemicals A and B?

Use A and B since they relate to the problem statement (any symbols could be used, X and Y, tor example).

Is there a simple way to express one variable in terms of the other, so we can deal with only one variable?

I don't see how. Let's keep both A and B.

What conditions are we asked to satisfy?

That the total amount of nickel must be 50 grams and the total amount of chlorine must be 60 grams.

Where does the nickel and chlorine come from?

They come from chemicals A and B.

What relation would account for the total amount of nickel?

(fraction of nickel in A) x (amount of A) + (fraction of nickel in B) x (amount of B) = (total amount of nickel).

Using variables and constants (numbers), how can this equation be written?

(2/3)A + (1/4)B = 50 [1]

Write a similar equation for chlorine.

(1/3)A + (3/4)B = 40 [2]

We have reached our goal of transforming a word problem into an algebra problem. To be sure the student understands the methods of solving equations [1] and [2], let's go through their solution:

Fractions generally complicate the solution of problems. How can we get rid of tractions?

Multiply by a common denominator.

What is a common denominator for equation [1] and for equation [2]?
12 and 12 (Usually they are not the same values for other sets of equations. Just find each one separately.)
Multiply both [1] and [2] by this common denominator, 12.
8A + 3B = 600 [3] and 4A + 9B = 480 [4].

Now, we will solve [3] and [4] using two different methods:

Substitution method:

Solve an equation for a variable. Say, equation [3] for B.
3B = 600 - 8A, B = 200 - (8A/3) [5]
Substitute [5] into [4] to eliminate B from [4].
4A +9(200 - (8A/3)) = 480, 4A + 1800 -24A = 480, -20A = -1320, A = 66.
Now substitute this value of A into [5] to find B.
B = 200 - (8 x 66/3), B = 200 - 8 x 22, B = 200 - 176, B = 24.

Linear combination method:

What can we multiply equation [3] or equation [4] by to make the coefficient (multiplier) of either A or B the same or one the negative of the other?
Multiply [3] by 3 to make the coefficient of B in equation [3] equal the coefficient of B in equation [4]: 24A + 9B = 1800 [6]
How can [6] be used to eliminate B from equation [4]?
By subtracting [4] from [6] (or [6] from [4]):
(24A + 9B) - (4A + 9B) = 1800 - 480
What does that give?
20A = 1320, A = 66 [7]
Now, how can [7] be used to find B?
Substitute into an equation ([3] for example) to eliminate A and get an equation in only the remaining variable, B.
What results when you do that substitution?
8 x 66 + 3B = 600, divide by 3: 8 x 22 + B = 200, B = 200 -176, B = 24.

So, by either method, 66 grams of the first chemical should be combined with 24 grams of the second chemical.

The following table was constructed using the answers to the questions used to generate equations [1] and [2]. Add the A row to the B row to equal the C row in the last two columns to get equations [1] and [2].

Chemical	Weight	Fraction Nickel	Fraction Chlorine	Grams Nickel	Grams Chlorine
A	A	2/3	1/3	2A/3	A/3
B	B	1/4	3/4	B/4	3B/4
C	A + B			50	40

4.2 Motion Problem

Some motion problems are easily formulated in two variables. When there is no known relationship between the two, we are not able to use a single unknown. Consider the following simple problem involving two unknowns:

Sue and Tom are flying a small plane from central California to southern California, a distance of 300 miles. Flying south they have a tailwind that allows them to make the trip in only 2 hours. Flying north the same wind is blowing so it takes them 3 hours to make the trip back. How fast would the airplane fly in still air (its airspeed)? How fast was the wind blowing?

What symbols do you want to assign to the unknowns?
Let the airspeed be A and the windspeed be W.
What is the relationship between distance, time and speed?
Distance = time x speed.
What is the speed with the wind?
Speed = A + W.
What is the speed against the wind?
Speed = A - W.
Can you set up a table showing time, speed and distance for this problem?

Direction	Time	Speed	Distance
South	2	A + W	300
North	3	A - W	300

Using the relation between time, speed and distance, can you write two equations using the table data?
2(A + W) = 300 [1] and 3(A - W) = 300 [2].

There is more than one way to solve equations [1] and [2]. The following is one way:

Can you simplify these equations by dividing by the multipliers of the quantities in parentheses?
A + W = 150 [3] and A - W = 100 [4].
What do you get when you add these two equations?
2A = 250 so, A = 125.
How can you find W?
Substitute 125 for A in equation [3]: 125 + W = 150, so, W = 25. The airspeed is 125 miles per hour and the windspeed is 25 miles per hour.
Do the answers check?
The speed going south was 125 + 25 = 150 miles per hour. So, the airplane could travel 300 miles south in 2 hours. The speed going north was 125 - 25 = 100 miles per hour. So, the airplane could travel 300 miles north in 3 hours. The answers check.

4.3 Number Problem

Some more complicated number problems can more easily be solved by introducing two variables rather than trying to express them in terms of a single variable.

If the numerator of a fraction is decreased by 1 and its denominator is increased by 3, the resulting fraction is 1/4. If the numerator of the original fraction is increased by 1 and the denominator is decreased by 3, the resulting number is 2. What is the original fraction?

Instead of a single unknown, what two unknowns do we want to find?
The numerator (call it N) and the denominator (call it D). Original fraction = N/D.
Can you write an equation using the first sentence of the problem?
$(N - 1)/(D + 3) = 1/4$. Get rid of fractions by multiplying by $4(D + 3)$ the common denominator: $4(N - 1) = (D + 3)$. Remove the parentheses: $4N - 4 = D + 3$.
Move the variables to the left and constants to the right of the equal sign:
$4N - D = 7$ [1]
Now, can you write a similar equation based on the second problem sentence?
$(N + 1)/(D - 3) = 2$. "Undo" the division by multiplying by $D - 3$: $N + 1 = 2(D - 3)$.
Remove parenthesis and move variables to the left and constants to the right of the equal sign: $N - 2D = -7$ [2]
Can you solve equations [1] and [2] by substitution?
Solve [2] for N: $N = 2D - 7$ [3]. Substitute this into equation [1]:
$4(2D - 7) - D = 7$, $8D - 28 - D = 7$, $7D = 35$, $D = 5$. Substitution of 5 into equation [3] gives: $N = 3$.
So, the original fraction is 3/5.
Can you solve equations [1] and [2] by linear combination?
Multiply [1] by 2: $8N - 2D = 14$. Subtract equation [2] from this equation:
$(8N - 2D) - (N - 2D) = 14 - (-7)$, $7N = 21$,
$N = 3$. Substitute 3 into equation [2] for N: $3 - 2D = -7$, $-2D = -10$, $D = 5$ as before.
Does your answer check?
$(3 - 1)/(5 + 3) = 2/8 = 1/4$. $(3 + 1)/(5 - 3) = 4/2 = 2$. Yes.

5. Word Problems Involving Cooperation and Opposition:

Some word problems seem very difficult to the beginner until he understands one simple idea. An example of this type of problem involves cooperation between workers. The simple idea is to express the fraction of a job each participant can complete per unit time.

5.1 Business Problems Involving Cooperation

Consider the following problem:

Tom can plant a field of corn in 6 days. Jim can plant the same field of corn in 8 days. How many days will it take them working together to plant the field of corn?

The following dialog will illustrate the power of expressing this problem using fractions.
What does the problem ask us to find?

The number of days to do the job.
What do you want to call the unknown?
Let's call it "D" (or X or any symbol).
What part of the job can Tom do in 1 day?
He can do 1/6 th of the job.
What part of the job can Tom do in "D" days?
He can do D/6 th of the job.
What part of the job can Jim do in "D" days?
He can do D/8 ths of the job.
If we add the fractions of the job that each can do, what do we get?
We get the whole job. That is, 1 job.
Write the resulting equation.
$D/6 + D/8 = 1$ [1]

This equation can be solved by the principles described above.
How can the fractions (divisions) be "undone"?
By multiplying by a common denominator.
Multiply [1] by its lowest common denominator.
The lowest common denominator is 24, so $4D + 3D = 24$ [2]
Combine like terms and solve for D.
$7D = 24$, $D = 3$ and 3/7 days.

A small table can be generated for this problem:

Number of days	working alone	Fraction per day	Number of days	Total part
Tom	6	1/6	D	D/6
Jim	8	1/8	D	D/8
Both			D	1

From the last column, we can write: $D/6 + D/8 = 1$ the same as equation [1].

Once the power of using fractions to solve this type of problem is mastered, the method can easily be applied to solve more complex problems than the one above. For example:

Two pipes flowing simultaneously can fill a container in 4 hours. One by itself could fill the same container in 6 hours. How long would it take he second pipe alone to fill the container?

Let us use fractions to solve this problem.
What shall we call the number of hours it takes the second pipe alone to fill the container?
Let us call it "H".
What fraction of the container can be filled per hour by both pipes together? By the first pipe alone?
Both pipes 1/4 th. First pipe alone 1/6 th.

What fraction of the container can be filled by the second pipe in one hour?
1/H.
Form an equation relating the three fractions.
1/6 + 1/H = 1/4 [1]
What is the common denominator of [1]?
12H.
Multiply by that to remove divisions.
2H + 12 = 3H, 12 = H, So, it would take 12 hours.

Some students like to organize the problem solution by setting up a table like the following:

	Days to fill	Fraction filled per day
First pipe	6	1/6
Second pipe	H	1/H
Both pipes	4	1/4

Using the last column, add the first two rows and set the sum equal to the last row. This will result in equation [1].

In some problems not all "actors" perform for the total elapsed time. The following problem is an example:

Tom, Dick and Harriet work in a bakery. Tom would take 4 hours to bake enough bread to fill a delivery van. Dick could do the same amount in 6 hours. Finally, Harriet would need 5 hours to do the job. Harriet came to work at 6 AM but had to leave at 7 AM . At that time, Tom and Dick arrived then finished filling the van. When were they through?

Who are the "actors" in this problem?
Tom, Dick and Harriet.
What do we know about these "actors' characteristics" that will help us solve this problem?
What fraction of the job they do in a given amount of time.
How long did Harriet work?
One hour (note: do not use clock times - use the elapsed time).
What should the variable be for this problem?
The length of time Tom and Dick worked (do not use clock times until the first equation has been solved). Let us call it "t".
Other than the fraction of the job done per hour. What other quantities will help us solve this problem?
The number of hours worked and the part of the job completed in that time.
Can you construct a table organizing this information?

Person	Fraction/hour	Number of hours	Fraction done
Tom	1/4	t	t/4
Dick	1/6	t	t/6
Harriet	1/5	1	1/5
All			1

Can you write an equation that will allow you to solve for "t"?
From the last column: t/4 + t/6 + 1/5 = 1. [1]
How can this equation be simplified?
By removing fractions. This can be done by multiplying by a common denominator.
What is the least common denominator?
2 x 2 x 3 x 5 = 60. Note: a common denominator is 4x6x5, but 4 and 6 both contain 2 as a factor, so one 2 can be eliminated for the least common denominator.
After multiplying by 60, what does equation [1] become?
15t + 10t + 12 = 60.
Can you solve this equation for "t"?
Get rid of the 12 term by subtracting 12 from both sides. Combine like terms:
25t = 48. "Undo" the multiplication by dividing by 25: t = 1 23/25 hours. That is, a little less than two hours. The exact time interval is 1 hour plus (23/25) x 60 minutes/hour = 1 + 55.2 minutes = 1 hour + 55 minutes + .2x60 seconds/minute = 1 hour, 55 minutes and 12 seconds.
Is this the answer to the problem?
No. The problem asks what time Tom and Dick will be through. It will be 7 AM plus almost 2 hours. That would make it just before 11 AM. Or, exactly 10:55:12 AM.

5.2 Motion Problem Involving Dpposition

When two agents are acting in opposition, we must subtract their contributions instead of add them. The following problem illustrates this distinction:

Jane and Kathy are on the high school track team. Jane can go around the oval track in 5 minutes. Kathy takes 5.5 minutes. How long will it be until Jane "laps" Kathy?

Assume Jane and Kathy can keep running as long as necessary. Note that we do not need to know the length of the track. Again, the key to this problem is to find the fraction of a lap each person runs per minute.

What fraction of a lap can Jane run in 1 minute? What fraction can Kathy run in 1 minute?
Jane: 1/5. Kathy: 1/5.5.
What symbol do you want to use for the unknown?
Since we want to find the time for Jane to "lap" Kathy, let T represent that time.
Can you construct a table that organizes the problem solution?

Person	Fraction/minute	Time	Laps
Jane	1/5	T	T/5
Kathy	1/5.5	T	T/5.5
Jane - Kathy		T	1

Does the last column of this table suggest the appropriate equation?
Yes. Laps run by Jane - Laps run by Kathy = 1. In symbols: $T/5 - T/5.5 = 1$.
Can you solve this equation?
Multiply by the common denominator, (5)(5.5): $5.5T - 5T = 27.5$. Collect like terms: $.5T = 27.5$. Divide by .5: $T = 55$ minutes.
Does your answer check?
In 55 minutes, Jane can run 55/5 that is 11 laps. Kathy can only run 55/5.5 that is 10 laps. So, it does check.

6. Maximum and Minimum Problems:

One type of word problem causes some students to give up without a fight. They don't see how algebra can be used to answer questions like "how can some quantity be made the largest or smallest possible?". For example, consider the following problem:

A farmer needs to fence in an area to protect his carrot patch from rabbits. She has only 60 feet of fencing to do the job. She wants a rectangular patch with the largest possible area so she can grow as many carrots as possible. Prove (don't just guess), using algebra, what the length and width of the patch should be.

6.1 Finding Minima and Maxima

Before attacking the farmer's problem, let's understand how a minimum value can be found.

When a number is squared, will the result be positive, negative or zero?
Either positive or zero, never negative.
If the square of a number is added to a second number, what value of the first number will produce the smallest sum? [A]
Zero, since any other value will increase the sum.
From this, what general formula will result in a minimum?
Minimum = (some function of the variable)^2 + constant, where (some function of the variable) = 0 [1]

In a similar way, we can find the largest value of a function. Just change statement [A] to:

If the negative of the square of a number is added to a second number, what value of the first number will produce the largest sum?

Zero, since any other value, positive or negative, will increase the sum and make the negative of the sum more negative.
From this, what general formula will result in a maximum?
Maximum = - (some function of the variable)^2 + constant, where (some function of the variable) = 0 [2]

6.2 Completing the Square

Unfortunately, often the function of the variable mentioned above is not in the proper form. We will use a method called "completing the square" to convert an equation of the form $X^2 + bX + c$ [3] where b and c are constants (positive, negative or zero) to the form like the one needed above $(X + d)^2 + e$ [4] where d and e are also constants.

If two expressions are equal (like [3] and [4] are to be made equal), what can you say about the coefficients of like terms?
They must be equal.
What does [4] become when (X + d) is squared?
$X^2 + 2dX + d^2 + e$ [5]
Compare [5] and [3]. Set the coefficients (multipliers) of X equal.
$2d = b$ [6]
So if we know b from equation [3], what is d in equation [4]?
$d = b/2$ [7]
Compare [5] and [3]. Set the constants equal.
$d^2 + e = c$ [8]
Substitute [7] into [8] and solve for e.
$(b/2)^2 + e = c$; $e = c - (b/2)^2$ [9]
Using [7] and [9], how can [4] be written in terms of b and c?
$(X + b/2)^2 + (c - (b/2)^2)$ [10]
Check your work by squaring the term (X + b/2).
$X^2 + bX + (b/2)^2 + c - (b/2)^2 = x^2 + bX + c$ just like [3]!
Why is this method is known as "completing the square"?
Because $X^2 + bX$ will not be the square a function like X + d unless we add $(b/2)^2$ to it (and make d = b/2). Since we add $(b/2)^2$ to $X^2 + bX$, we must also subtract it to avoid changing the original $X^2 + bX$ function.

6.3 Top-Down Solution

Now let us use the top-down method to solve the farmer's problem.
What is the most general statement we can make?
We know that we need the area of a rectangle as a function of its length and width.
Area = length x width. [11]
We don't know either the length or width, so we need to choose at least one to

be the variable. Which should it be?

Let "L" be the length (it doesn't matter, we could have chosen the width; we could use "X" or any letter for the symbol).

What should we use for the width? (It is usually best to introduce as few variables as possible when other quantities can be expressed simply in terms of previously chosen variables). Other than the area, what connects the length and width of a rectangle?

Perimeter = 2(length + width).

Using the problem statement and our variable, how can this equation be rewritten?

60 = 2 x (L + width); or, dividing by 2: 30 = L + width; solving for width:

width = 30 - L [12]

Using equation [12] and the symbol for length, how can equation [11] be expressed?

Area = L x (30 - L) [13]

What needs to be done now is to cast equation [13] into the general form for finding the maximum value. Do you remember what that form is?

Maximum = - (some function of the variable)^2 + constant, where (some function of the variable) = 0. [2]

How can equation [13] be transformed to look more like [2]?

Area = - (L^2 - 30L); then completing the square: Area = - (L^2- 30L + 225) + 225

Remember from the last sentence of section 6.3, to complete the square we take one-half of the coefficient of L, square it and add and subtract this square to the equation. From the last equation: Area = - (L - 15)^2 + 225.

So, what is the function of the variable L referred to in equation [11] for this problem?

L - 15.

What condition must this function satisfy for maximum area?

L - 15 = 0. That is, L = 15.

What is the width of the rectangle?

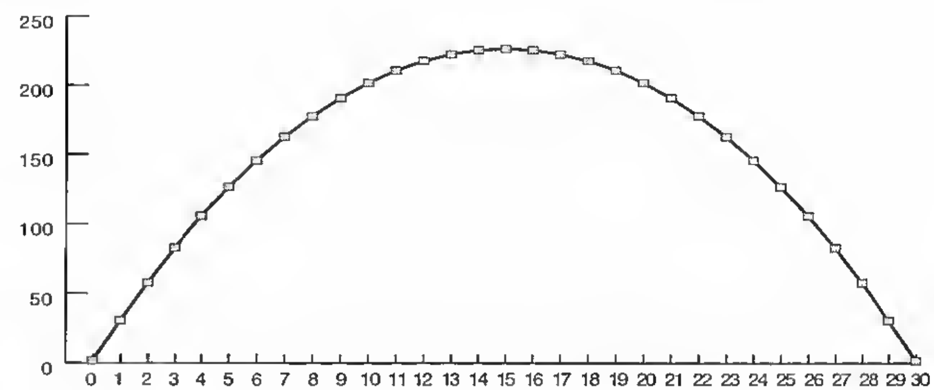
From equation [12]: width = 30 - L. So, width = 30 - 15 = 15.

Since the length and width are equal, the carrot patch will be square.

The explanation of how to attack this problem is long. Several concepts are involved and we want the student to understand each of them. Note, when a student understands how a solution or method such as completing the square is derived, it is easier to remember. Or, if forgotten, it can be derived by the student.

6.4 Graph of Maximum

I have drawn a graph below showing how the area of the carrot patch would change as the length of one side changes. Other problems dealing with maximum values will have a similar shape. For minimum problems, the curve would be inverted to look like a bowl.



Graph of the Area of the Carrot Patch

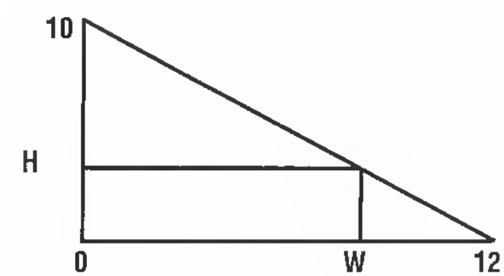
The student may wonder how to treat a function like $aX^2 + bX + c$ when completing the square. Probably the simplest way is to factor out the "a": $a(X^2 + b'X + c')$ where $b' = b/a$ and $c' = c/a$. Then use the method described above to get: $a(X + b'/2)^2 + c' - (b'/2)^2$

6.5 Additional problem

Here is an additional problem asking for a minimum result. I will state the problem. Then give some hints as how to proceed. First, try to solve the problem without referring to the hints or as few hints as possible. Finally, I will give an abbreviated version of the solution.

What is the area of the largest rectangle that can be inscribed in a right triangle with legs that are 10 and 12 inches long?

Hint: Draw a diagram that illustrates the conditions of the problem. This is usually very worth-while.



What should be our basic variable?

Chose the rectangle width, W, as the variable. Express the rectangle height, H, in terms of W.

Hint: The hypotenuse (line that connects the ends of the sides (legs) that form a right triangle) is also the graph of H as a function of W. Use the slope, y-intercept form of the line to express H as a function of W.

Slope is -10/12, y-intercept is 10. So the equation of H(W) is $H = 10 - (10W)/12$

or $H = 10 - (5W)/6$ [14]

What is the area of a rectangle of width, W, and height, H?

Area = $H \times W = (10 - (5W)/6)W$

What form should the equation for area be in to find a maximum?

Like equation [2]: Area = - (function of variable)^2 + constant. For this problem:

$$\text{Area} = -(5W^2 - 60W)/6, \quad \text{Area} = -5(W^2 - 12W)/6$$

Note: If Area is a maximum, so will be 6xArea/5. Thus we can ignore the 5/6 th factor.

Complete the square for $-(W^2 - 12W)$.

$-(W^2 - 12W + (-6)^2) + (-6)^2 = -(W - 6)^2 + 36$

Now what is the W that will give the maximum area?

Set $W - 6 = 0$, so $W = 6$ and from [14] $H = 10 - 5(6)/6 = 10 - 5$, $H = 5$.

7. Word Problems Involving Inequalities

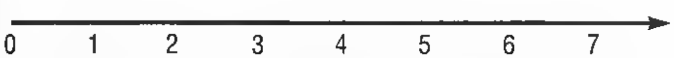
7.1 Inequality Rule and the Number Line

An inequality is a relation that expresses "less than (<)," "less than or equal (≤)," "greater than (>)," or "greater than or equal (≥)" rather than simple equality. Treating equations involving inequalities is simple. We need add only one new rule. Once a student is confident he or she understands this rule, problems involving inequalities need be no more difficult than similar problems involving equalities. The key principle is:

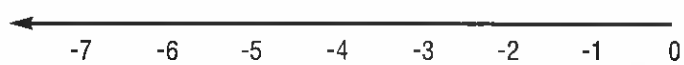
Multiplying or dividing by a negative number reverses the inequality.

For example: $7 > 5$. If we multiply by -2, we get: $-14 < -5$. Division is in the rule, since division can be thought of as multiplication by a fraction.

I think that the easiest way for a student to understand the behavior of negative numbers and what happens when multiplying by negative numbers is graphically. Let us represent the positive numbers by a line starting at zero and proceeding to the right:



Think of multiplication by -1 as rotating this line by 180 degrees:



A number to the left of another is less than the one to the right. A number to the right of another is greater than the one to the left. Thus, we see how multiplying by -1 reverses an inequality. Multiplying by a number other than 1 or -1 also expands or shrinks the number lines. Of course, this idea of a number line is just a fiction. However, it helps us visualize some numerical operations. For example, we can visualize the strange equality of $-(-1) = 1$ as a rotation of the left pointing arrow "-1", 180 degrees to point in the positive (right) direction.

7.2 Simple Inequality Word Problems

Usually it is best to formulate the equations to solve an inequality word problem as if the problem involved equalities. Then, once the equation is derived, replace the equal sign (=) by the appropriate inequality sign (< or >). For example:

John's employer will provide medical insurance from either of two insurers. With plan X, John would have to pay the first $200 of medical costs and the plan would pay 70% of the remaining expenses. With plan Y, John would have to pay the first $300 of medical costs and the plan would pay 90% of the remaining expenses. For what total medical bills would plan X be the better plan?

Eventually, we must form an inequality involving the costs of both plans. For now, let's form an equation that equates the plan costs.

What variable is in both plan costs?

Total year's medical bills. Let us call the total, B.

What would be the cost of plan X?

Cost = $200 + (1 - .7)(B - 200)$ [1] (the plan pays 70% so John pays 100% - 70%)

What would be the cost of plan Y?

Cost = $300 + (1 - .9)(B - 300)$ [2] (the plan pays 90% so John pays 100% - 90%)

When would plan Y be better than plan X?

When the cost [1] is less than cost [2].

Write an inequality relating equations [1] and [2].

$300 + .1(B - 300) < 200 + .3(B - 200)$

How can this be simplified?

Multiply by 10 to remove decimals: $3000 + (B - 300) < 2000 + 3(B - 200)$; Remove parenthesis so we can collect like terms: $3000 + B - 300 < 2000 + 3B - 600$; $B + 2700 < 3B + 1400$; Subtract 3B and subtract 2700 to move the variable to the left and constants to the right: $-2B < -1300$; Divide by -2 to remove the coefficient of B (set it to 1): $B > 650$.

Note that the inequality switched from less than (<) to greater than (>) when the inequality is divided by a negative number (-2). Thus, plan Y will be the better when the medical bills are larger than $650.

8. Solution of Quadratic Equations by Factoring

8.1 How to Factor Quadratic Equations

Quadratic equations are equations of the form: $aX^2 + bX + c = 0$ [1], where a, b and c are constants and a is not 0. In factored form, the equation will have the general appearance of $(dX + e)(fX + g) = 0$ [2], where d, e, f and g are constants. Let us see how we can get equation [1] from equation [2].

How can we get equation [2] to have an X^2 term like equation [1] has?

When [2] is multiplied out, we get: $dfX^2 + (dg + ef)X + eg = 0$ [3]

So, what in equation [1] corresponds to "df", the coefficient of X^2, in equation [3]?
"a" corresponds to "df", that is: df = a so, "d" and "f" are factors of "a".
What in equation [1] corresponds to "eg", the constant term in equation [3]?
"c" corresponds to "eg", that is: eg = c so, "e" and "g" are factors of "c".
What in equation [1] corresponds to "dg + ef", the coefficient of X in equation [3]?
"b" corresponds to "dg + ef", that is: dg + ef = b.

So, a method of factoring many quadratic equations is the following:

Find pairs of factors of "a", the coefficient of X^2. ("1" and "a" are one such pair).
Find pairs of factors of "c", the constant term. ("1" and "c" are one such pair).
Multiply one factor from "a" times one factor from "c". Multiply the other factor of "a" times the other factor of "c". Add the two products.

If the preceding sum equals "b", the coefficient of X in the quadratic equation, you have found the factors of the quadratic equation [1] as defined in [2].

The student need not memorize this process, just understand how it was developed. A simple example will illustrate factoring a quadratic equation:

$X^2 + 5X + 6 = 0$ [4]

What are factors of the coefficient of X^2?
"1" and "1" since the coefficient is "1" (understood).
What are factors of the constant, "6"?
"1" and "6", "2" and "3".
Try factors $(X + 1)(X + 6)$.
These multiply giving: X2 + 7X + 6 this does not result in "5" for "b".
Try factors $(X + 2)(X + 3)$ [5]
These multiply giving: $X^2 + 5X + 6$ this does result in "5" for "b". So, [5] are factors of [4].
A more complicated example is:
$6X^2 - X - 15 = 0.$ [6]
What are all pairs of factors of 6?
1 and 6, 2 and 3.
What are all pairs of factors of 15?
1 and 15, 3 and 5.

What does a negative "c" signify (c = -15)?
That one and only one of the factors of 15 must be negative.
Try all the combinations of factors until you find the correct one. What is it?
$(X + 1)(6X - 15)$ No. $(X - 15)(6X - 1)$ No. $(X - 5)(6X + 3)$ No. $(X + 3)(6X - 5)$ No.
$(2X + 1)(3X - 15)$ No. $(2X - 15)(3X + 1)$ No. $(2X - 5)(3X + 3)$ No. $(2X + 3)(3X - 5)$ Yes!

Note that if we try $(X + 1)(6X - 15)$, we won't have to try $(6X - 15)(X + 1)$ since, in general:
a x b = b x a (commutative law of multiplication).
So, $(2X + 3)(3X - 5) = 0$ is the factored form of [6]. If ab = 0, what does that tell you about a and b?
Either a = 0, or b = 0, or both a and b = 0.
When you set the factors of [6] to zero, what do you get for X?
$2X + 3 = 0$, $2X = -3$, $\mathbf{X = -3/2}$ and $3X - 5 = 0$, $3X = 5$, $\mathbf{X = 5/3}$.

8.2 Quadratic Formula

In section 6.2 we learned a method called "Completing the Square". This method can be used to solve any quadratic equation that has a "real" solution. Later, we will discuss what is meant by a "real" solution. To review, divide equation [1] by "a", to get:
$X^2 + (b/a)X + (c/a) = 0$. Take half the coefficient of X, square it, and add it to both sides of the equation. $X^2 + (b/a)X + (b/(2a))^2 + (c/a) = (b/(2a))^2$. Factor the first three terms: $(X + b/(2a))^2 + (c/a) = b^2/(4a^2)$. Now, move the constant term to the right-hand side by subtracting c/a from both sides and take the square root:
$X + b/(2a) = \pm\text{sqrt}(b^2/(4a^2) - c/a)$, or, $X = -b/(2a) \pm\text{sqrt}((b^2 - 4ac)/4a^2)$.
Finally, we get the quadratic formula: $\mathbf{X = (-b \pm sqrt(b^2 - 4ac))/(2a)}$ [6]
The student can memorize this formula to save time, however the important thing is to understand how it was derived. The expression $b^2 - 4ac$ is called the discriminant because it discriminates between the nature of the various types of roots. If $b^2 - 4ac$ is positive, not zero, there are two roots. If $b^2 - 4ac$ is zero, there is only a single root: $-b/(2a)$. If $b^2 - 4ac$ is negative, we cannot take a square root, so there are no "real" roots. The roots are called "imaginary". Advanced algebra courses consider the properties of these "imaginary" roots. Let us use formula [6] to solve equation [5].
What are a, b and c for equation [5]?
a = 6, b = -1, c = -15.
Substitute a, b and c for equation [5] into equation [6].
$X = (-(-1) \pm\text{sqrt}((-1)^2 - 4(6)(-15)))/(2(6)) = (1 \pm\text{sqrt}(1 + 360))/12 = (1\pm\text{sqrt}(361))/12$
$X = (1 \pm 19)/12 = -18/12$ or $20/12 = -3/2$ or $5/3$.
This is the same result as we got before.

8.3 Number Problems

When two numbers are multiplied, a quadratic equation often results. For example:
A certain number is 2 less than twice a second number. When the two numbers are multiplied, their product is 24. What are the numbers?

What will be your variable?
Since the first number is a function of the second number, let's call the second number, S.
What is the first number?
2S - 2.
Can you derive an equation from the second sentence?
$(2S - 2)S = 24$, $2S^2 - 2S - 24 = 0$.
Can this equation be factored?
Yes. $(2S + 6)(S - 4) = 0$.
Set both factors to zero. What are the solutions?
First solution: $2S + 6 = 0$, $2S = -6$, $S = -3$ and the first number is $2(-3) - 2 = -8$.
Second solution: $S - 4 = 0$, $S = 4$ and the first number is $2(4) - 2 = 6$. Thus, either -8 and -6 or 6 and 4 satisfy the conditions of the problem.

Another way a quadratic equation may be generated is when a number and its reciprocal are both in a relation (the reciprocal of N is 1/N). For example:
When the reciprocal of a number is subtracted from 6 times the number, the result is 5. What Is the number?
What symbol do you want to assign to the number?
Call it N.
Can you write an equation involving N based on the problem statement?
$6N - 1/N = 5$.
Can you convert this equation to the standard quadratic form $aX2 + bX + c = 0$?
Multiply by N to remove the divisor: $6N^2 - 1 = 5N$, rearrange terms: $6N^2 - 5N - 1 = 0$.
What are factors of this equation?
$(6N + 1)(N - 1) = 0$.
Set each factor to zero. What can N be?
$6N + 1 = 0$, $6N = -1$, $N = -1/6$ or $N - 1 = 0$, $N = 1$. Both -1/6 and 1 are solutions.
Do these answers check out?
$6(-1/6) - 1/(-1/6) = -1 - (-6) = -1 + 6 = 5$. $6(1) - 1/(1) = 6 - 1 = 5$. Yes.

8.4 Digit Problem

Digit problems are similar to number problems in that when digits are multiplied, quadratic equations can result. For example:
The unit's digit of a two digit number is half the ten's digit. If the number is multiplied by the sum of its digits, the product is 63. What is the number?
What should the variable be? What symbol do you want to use?
The unit's digit is a function of the ten's digit. However, since the ten's digit is twice the unit's digit, use the unit's digit to avoid a fraction. Let's call it U.
What is the ten's digit? What is the number?
Ten's digit = 2U. Number = $10(2U) + U = 21U$.
Using the problem's second sentence,

can you construct an equation?
Number x sum of digits = 63. $21U(2U + U) = 63$, $63U^2 = 63$, $U^2 = 1$, $U = 1$ (ignore -1 since it is not a digit.)
What is the number?
Ten's digit is $2U = 2(1) = 2$. The number is 21.
Does your answer check?
$21 \times (2 + 1) = 21(3) = 63$. Yes.

8.5 Motion Problems

Don't forget to make sure units of measurement are consistent. Motion problems can generate quadratic equations. As the following problems show:
John's boat can go 12 miles per hour in still water. He travelled 36 miles upstream and 36 miles downstream in 8 hours. How fast was the current flowing?
What symbol do you want to use for the unknown?
Use C for the current's speed.
Can you make a table showing the problem data?

Trip conditions	Speed	Distance	Time
Upstream	**12 - C**	**36**	**36/(12 - C)**
Downstream	**12 + C**	**36**	**36/(12 + C)**
Round trip		**72**	**8**

What is the relation that connects the three times shown in the last column?
Upstream time + downstream time = total time. $36/(12 - C) + 36/(12 + C) = 8$.
Can you solve this equation for C?
Multiply by the common denominator to "undo" divisions:
$36(12 + C) + 36(12 - C) = 8(12 - C)(12 + C)$, divide by 4 to simplify coefficients:
$9(12 + C) + 9(12 - C) = 2(12 - C)(12 + C)$, multiply to remove parentheses:
$108 + 9C + 108 - 9C = 2(144 - C^2)$, $216 = 288 - 2C^2$, $2C^2 = 72$, $C^2 = 36$.
$C = \pm 6$. From the way the problem was stated, as upstream and downstream, the current has to be positive. So, the current flows at 6 miles per hour.
Does this answer check?
$36/(12 + 6) + 36(12 - 6) = 36/18 + 36/6 = 2 + 6 = 8$. Yes.

Try to solve this more complicated motion problem:
A race car driver is entering a race over a 120 mile course. It he can increase his average speed by 20 mph, he could save 5 minutes and possibly win the race. What is his average speed now?
Are units ot measurement consistent?

No. I will convert the time saved from minutes to hours: 5 minutes = 1/12 hour.
What do you want to call the unknown speed?
Call his current average speed S.
What would be the increased speed?
S + 20.
How long will it take to travel 120 miles at the two different speeds?
Time = distance/speed. Current speed: time = 120/S. Increased speed:
time = 120/(S + 20).
Using these two times, can you write an equation for the time saved?
120/S - 120/(S + 20) = 1/12.
Can you convert this equation to the form: $aX^2 + bX + C = 0$?
Multiply by the common denominator: 12S(S + 20): 1440(S + 20) - 1440S = S(S + 20).
Remove parenthesis: 1440S + 28800 - 1440S = S^2 + 20S. Collect like terms and rearrange: $S^2 + 20S - 28800 = 0$.
Can you factor this equation?

Hint: since the b of the standard form is positive (20) and the c is negative (-28800), one factor of 28800 must be negative and the other positive. Also, the negative factor has an absolute value (magnitude ignoring its sign) that is only 20 less than the positive factor because the coefficient of S is 20. This means that S^2 is approximately the square root of 28800. Since this square roof is approximately 170, try factors near 170 and 20 apart in absolute value.

(S + 180)(S - 160) = 0.
What is the current speed?
Set the first factor to zero: S + 180 = 0, S = -180 (unlikely that he can backup that fast!). Set the second factor to zero: S - 160 = 0, S = 160 miles per hour.
Does this answer check out?
Time at 160 mph = 120/160 = 3/4 hour. Time at 180 mph = 120/180 = 2/3 hour.
Time saved = 3/4 - 2/3 = 9/12 - 8/12 = 1/12 hour or 5 minutes. Yes, it checks.

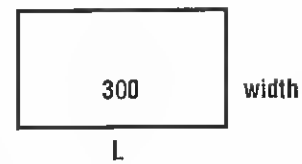
8.6 Geometry Problems

Problems involving area usually generate a quadratic equation since a length is squared or two lengths are multiplied. For example:

Jill has 70 feet of fencing to enclose a rectangular vegetable garden. The garden is to occupy an area of 300 square feet. What should be the dimensions of the garden?

What unknown quantities do we need to find?
The length and width of the rectangle.
Choose a symbol for one of these. Can you express the other as a function of this symbol?
Let L represent the length. Since the perimeter = 2(length + width),
70 = 2(L + width),
35 = L + width, so, width = 35 - L.

What is the area?
Length x width = area, L(35 - L) = 300



$35L - L^2 = 300$, $L^2 - 35L - 300 = 0$.
Can you factor this quadratic equation?
(L - 15)(L + 20) = 0. So L can be either 15 or 20 feet.
Using these possible values, what would the width be?
Width = 35 - L. So, the width would be either 20 or 15 feet. The rectangle should be 20 feet by 15 feet using either set of answers!

This is often the case for geometric problems when both answers are positive. Both results may be answers to different aspects of the problem. In this problem, the dimensions are 20 by 15 or 15 by 20: both solutions are equivalent.

Many word problems involve factoring. This problem also illustrates a problem with two numerical solutions, but only one is geometrically possible.

What three consecutive integers could be the lengths of the sides of a right triangle?

What do you know about the lengths of the sides of a right triangle?
The sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse (Pythagorean Theorem - see section 3.5).
Which side is longest, a leg or the hypotenuse?
The hypotenuse, because it has to reach from the end of one leg to the end of another going at right angles to the first. The Pythagorean Theorem also proves this.
What shall we call the shortest leg (the smallest integer)?
Let us call it "s" for shortest.
Then, what would the other sides be?
The other leg would be s + 1. The hypotenuse would be s + 2.
Write an equation using the Pythagorean Theorem.
$s^2 + (s + 1)^2 = (s + 2)^2$
What needs to be done to change this into a quadratic equation like $aX^2 + bX + c = 0$?
Square, remove parenthesis and collect terms: $s^2 + (s^2 + 2s + 1) = (s^2 + 4s + 4)$
$s^2 + s^2 + 2s + 1 = s^2 + 4s + 4$, $s^2 - 2s - 3 = 0$ [7]
Can you factor equation [7]?
The factors of "a" are 1 and 1. The factors of "c" are 1 and 3, and one of them must be negative. (s + 1)(s - 3) = 0 [8]
What is "s"?
Set the first factor to 0: s + 1 = 0, s = -1 a triangle cannot have a negative length!
Set the second factor to 0: s - 3 = 0, s = 3. O.K.
What are lengths of the three sides?
3, 4 and 5.

This problem also shows that there is only one set of three consecutive integers that can be the lengths of the sides of a right triangle.

The following problem also involves area:

An artist believes that a picture looks best when the frame has half the area of the picture. The picture to be framed is 40 inches high and 60 inches wide. How wide should the frame be?

What symbol do you want to uses to represent the frame width?

Use F to represent the frame width (as usual any symbol could be used).

How are the area of the frame, area of the picture alone and the area of the picture with frame related?

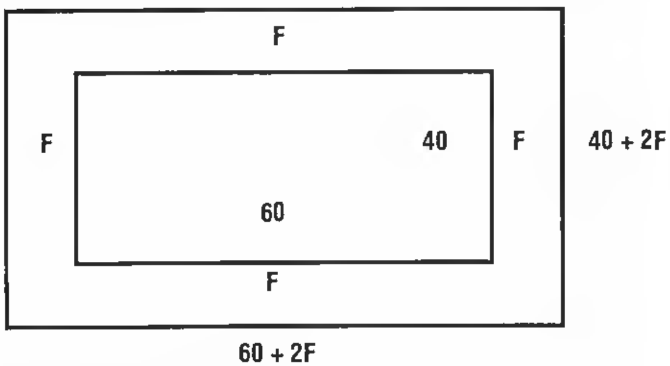
Area of frame = area of picture with frame - area of picture alone [1]

What is the area of the picture?

40 inches x 60 inches = 2400 square inches. [2]

Draw a diagram. What is the height and the width of the picture including the frame?

Height = 40 + 2F. Width = 60 + 2F.



What is the area of the picture with frame?

Area with frame = (40 + 2f)(60 + 2F) = 2400 +200F + $4F^2$ [3]

What is the area of the frame?

Since the area of the frame is to be half the area of the picture without the frame, it is: 1200 square inches. [4]

When [2], [3] and [4] are substituted into [1] what results?

$1200 = (2400 + 200F + 4F^2) - 2400$

Can you write this equation in "standard form" ($aX^2 + bX + c = 0$)?

$4F^2 + 200F - 1200 = 0$ [5].

Is there a common factor in all of these coefficients (a, b and c)?

Yes. Four is a common factor.

What does equation [5] become when this common factor is removed?

Since a factor is a multiplier, it can be removed by "undoing" the multiplication by dividing: $F^2 + 50F - 300 = 0$ [6].

Equation [6] cannot be factored with integer factors. Can you use the quadratic formula from section 8.2 to find its factors?

The quadratic formula is: $X = (-b \pm \text{sqrt}(b^2 - 4ac))/(2a)$. For this problem: a = 1, b = 50 and c = - 300. So, $F = (-50 \pm \text{sqrt}(2500 - 4(1)(-300)))/(2(1))$,

$F = (-50 \pm \text{sqrt}(2500 + 1200))/2 = (-50 \pm \text{sqrt}(3700))/2$.

Since the square root of 3700 is only approximately 60.8, the value of F will only be approximate. $F = (-50 \pm 60.8)/2$, $F = -25 \pm 30.4$, F = -55.4 or 5.4 since F cannot be negative, F = 5.4 inches approximately.

Does this answer check?

$(40 + 2(5.4))(60 + 2(5.4)) = (40+10.8)(60 + 10.8) = (50.8)(70.8) = 3597 \approx 3600$.

3600 - 2400 = 1200 = 2400/2. Yes.

8.7 Business Problems

Problems involving cooperative work can result in quadratic equations. This can happen when the time each worker would take to finish a job alone is to be found. The following problem is an example:

Jane and Sue working together can finish a job in 24 days. It takes Sue 20 more days than Jane takes to do the job alone. How long does it take each, working alone, to do the job?

What do you want for the basic unknown?

Since Sue's time to do the job is in terms of Jane's time, choose Jane's time working alone to do the job. Call it J.

What is Sue's time to do the job?

J + 20.

What fraction of the job does each complete per day working alone?

Jane: 1/J. Sue: 1/(J + 20).

Set up a table that organizes the quantities for this problem. What row labels ("actors") do you want to use? What column headings ("characteristics") do you want to use?

Person	Days working alone	Fraction/day	Days	Part done
Jane	J	1/J	24	24/J
Sue	J + 20	1/(J + 20)	24	24/(J + 20)
Both			24	1

Using the last column, can you write an equation appropriate to the problem?

24/J + 24/(J + 20) = 1.

Can you convert this to the quadratic form $aX^2 + bX + c = 0$?

Multiply by the common denominator, J(J + 20), to remove fractions:

24(J + 20) + 24J = J(J + 20). Remove parentheses: $24J + 480 + 24J = J^2 + 20J$.

Collect like terms: $J^2 - 28J - 480 = 0$.

What are factors of this equation?
$(J - 40)(J + 12) = 0$.
What are possible values of J?
Set each factor to zero: $J + 12 = 0$, $J = -12$ Impossible. $J - 40 = 0$, $J = 40$ days.
Does this answer check out?
$24/40 + 24/60 = 3/5 + 2/5 = 1$. Yes.

The following problem concerns retail pricing:
Betty bought a number of cans of soft drink for $14.40. By the time they had been drunk, the store had raised the price by 2 cents per can. At the higher price Betty could only buy 24 fewer cans than before for the same cost. How many cans were bought the first time?
What symbol do you want to use to represent the unknown?
Call the number of cans bought the first time C.
What was the price per can at first?
1440/C.
What was the price per can later?
1440/C + 2.
How many cans could Betty buy at the new price?
C - 24.
Can you write an equation relating the new price, the new number of cans and the total cost?
Price x number = cost, $(1440/C + 2)(C - 24) = 1440$.
Do you know how to convert this equation to standard form ($aX^2 + bX + c = 0$)?
Remove the division by multiplying by C: $(1440 + 2C)(C - 24) = 1440C$. Multiply to remove parentheses: $1440C + 2C^2 - 34560 - 48C = 1440C$. Collect like terms: $2C^2 - 48C - 34560 = 0$. Divide by 2: $C^2 - 24C - 17280 = 0$.
How would you factor this equation?
Hint: Use the same reasoning as was used in section 8.5. Since the b of the standard form is negative (-24) and the c is negative (-17280), one factor of 17280 must be negative and the other positive. Also, the negative factor has an absolute value that is only 24 more than the positive factor. This means that c^2 is approximately the square root of 17280. Since this square root is approximately 130, try factors near 130 and 24 apart in absolute value.
$(C + 120)(C - 144) = 0$.
Can you solve this equation?
Set each factor to zero: $C + 120 = 0$, $C = -120$ not valid. $C - 144 = 0$, $C = 144$.
Originally 144 cans were bought. So each cost 1440/144 =10 cents, originally.
See if you can check your answer?
New price = 10 + 2 = 12 cents. New number of cans = 144 - 24 = 120.
Cost = 12 x 120 = 1440. So, the answer checks. This is a solution that is correct even though it does not agree with typical grocery store prices.

About the Author

Bob Barton was born in 1929 in Dallas, Texas. He graduated from Southern Methodist University in 1950 with a Bachelor of Science degree in chemistry. He entered the Air Force, served in Korea and later at the Nevada nuclear test site. After leaving the Air Force in 1954, he worked as a civilian at Kirtland Air Force base using one of the first electronic computers. By 1958, Bob earned a Master of Science degree in physics from the University of New Mexico while working during the day at Kirtland.

In 1958, Bob accepted a job at the University of California-run Lawrence Livermore National Laboratory as a computational physicist. He worked there until retiring in 1985. During this time he wrote several articles that appeared in Lab publications. He also co-authored several articles in the "Astrophysical Journal" and wrote a chapter in the book *Numerical Astrophysics.*

Bob served as a volunteer math tutor during the years he worked at the Livermore Lab. After leaving the Lab, he volunteered as an aide in the Math Learning Center at Livermore High School. (Mr. Al Ofiesh, a dedicated and skillful teacher, pioneered and still directs this program of independent study.) Bob assisted there from 1985 to 1992 when he moved to Pleasanton, California. He currently volunteers as an aide at the Heritage Foundation Senior Day Care Program in Pleasanton. Bob has a brother, Lynn. He married June Martorano in 1953. They have a daughter, Sharon, and two sons, Brad and Douglas. They also have three grandsons, Robert, Thomas and Corey.

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